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Free groups II

Free groups of rank n as groups acting on trees

Proseminar spring semester 2017

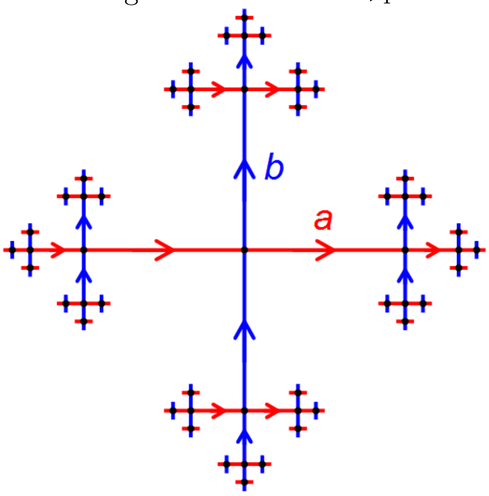
Contents

Chapter 1

Introduction

In this document we will give an introductory exposition about free groups. We will especially concentrate on a very known example, which is the free group $\mathbb{F}_2$. We will construct the Cayley graph of $\mathbb{F}_2$ (see Figure 1.1) and encounter some interesting facts about this group and its Cayley graph. Furthermore a very important topic will be actions of groups on graphs, especially on infinite trees. We will learn some facts about those actions and what we can construct "out of" free groups.

Figure 1.1: The tree $\mathcal{T}_4$



This tree $\mathcal{T}_4$ also corresponds to the Cayley graph of the free group $\mathbb{F}_2$ with basis S:=$\{a, b\}$.

Chapter 2

Free Groups

We begin this manuscript with the definition of finitely generated group. Later we will learn, how to construct free groups of finite rank using a specific equivalence relation. After introducing free groups, we will work with them and encounter interesting facts about them.

Definition 2.0.1. A group G is called **finitely generated**, if there exists a subset $S \subset G$, with $|S| < \infty$, such that the elements of S generate G, which we denote by $G =< S >$. S is called a generating set or a basis of G.

Definition 2.0.2. Let S be a set, which we call an alphabet. Then $\{S \cup S^{-1}\}^*$ is the set containing all possible words which are constructable by concatenation of elements of $S \cup S^{-1}$. The empty word is also an element of $\{S \cup S^{-1}\}^*$.

Example 2.0.3. Let S=$\{x, y, z\}$ be a set. Then $x^2x^{-1}y^2$ and x^3z^{17} are words in $\{S \cup S^{-1}\}^*$.

Definition 2.0.4. Let G be a group and $S \subset G$. A word $\omega \in \{S \cup S^{-1}\}^*$ is called **freely reduced**, if it does not contain any sub-word of the form aa^{-1} with $a \in S$.

This implies that the freely reduced form of the identity element is the empty word, since the identity is inverse to itself.
Using the seen information we can now define free generated groups.

Definition 2.0.5. A group is said to be a **free group with basis S**, if $S = \{x_1, ..., x_n\}$ is a generating set of G and no freely reduced word of the form x_i and its inverse represent the identity. This means in particular, that the identity element is not an element of the generating set S.

For a free group G with generating set S=$\{x_1, ..., x_n\}$, we denote the rank of G by:

$$rank(G) = |S| = n < \infty.$$

Furthermore, if G is free, we call G a free group of rank n and denote it by $\mathbb{F}_n$.

Theorem 2.0.6. *Given any $n \in \mathbb{N}$, there is a free group of rank n.*

Proof. Outline:
Let S=$\{x_1, ..., x_n\}$ be the set of n different and non-trivial elements. We introduce now the following equivalence relation on the set W:={S∪S^{-1}}* :

$$a_1 \dots a_i a_i^{-1} \dots a_k \sim a_1 \dots a_{i-1} a_{i+1} \dots a_k.$$

Here we need to mention that the words in W under this equivalence relation build equivalence classes, i.e. $[\mu]$ is the class containing the word $\mu \in W$ and $[\mu]$ is represented by the (freely) reduced version of μ. We aim now to define a multiplication on those equivalence classes in order to obtain a group.
But before defining the multiplication we need to show, that any equivalence class contains a unique reduced word. Here we refer to Jan, as he has already proved it in Proposition 2.0.5 of his talk.
We can conclude now, that each equivalence class contains a unique reduced word and define a multiplication on the set $\mathbb{F}_n$:={W/∼} by $[\alpha]$*$[\beta]$=$[\alpha\beta]$, where $\alpha\beta$ means the concatenation of the words α and β.
Now, we prove that ($\mathbb{F}_n$,*) is a group:
It is trivial, that the multiplication is associative, since the concatenation is associative. The identity element is the equivalence class $[\emptyset]$, where $\emptyset$ represents the empty word and the inverse of a class $[\alpha] = [a_1...a_k]$ is the class $[\alpha^{-1}]$=$[a_k^{-1}...a_1^{-1}]$. Since the equivalence classes contain a unique reduced word, the multiplication is well defined. Thus, ($\mathbb{F}_n$,*) is a group. It is especially a free group, since the empty equivalence class represents the identity and it is not contained in the generating set S. □

As we are studying free groups, we will need a definition of the length of words.

Definition 2.0.7. Let ω be an element of a free group of rank n with basis S. Since every element in a free group can be expressed by a unique reduced word, we define the **length** of ω as the number of letters of the alphabet S in the reduced version of the word. Notation: |g|:= length of g.

Above we have shown during the proof of Theorem 2.0.6, that the inverse of a freely reduced word is just its formal inverse. Therefore, we have that the lengths of ω and ω^{-1} are equal (i.e. $|\omega|$=$|\omega^{-1}|$).

Example 2.0.8. If we take the same word as previously used in an example we have: $x^2x^{-1}y^2$ is not a freely reduced word, but x^1y^2 is a freely reduced word. And the length is $|x^2x^{-1}y^2| = |x^1y^2| = 3$.

Concluding this section, we just have to remind ourselves, that freely reduced words in a **free group** are equal to reduced words. In the next chapter we will encounter an example of a free group and we will see some of its proprieties.

Chapter 3

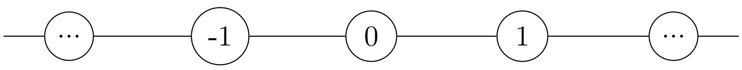
The free group $\mathbb{F}_2$ and some of its properties

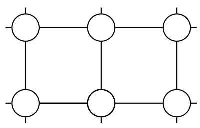
In this chapter we will concentrate on the free group $\mathbb{F}_2$. We will learn how its Cayley graph looks and how the free group acts on it.

Definition 3.0.1. Given a finitely generated group G with a symmetric generating set S (*i.e.* $S = S^{-1} \neq \emptyset$), the **Cayley graph** CAY(G), is the graph with vertex set V=G and edge set E=$\{(x, y) \in V \times V : \exists s \in S$ such that $y = xs\}$.

Definition 3.0.2. A **tree** is a connected graph without any circuits. We call a tree an **infinite tree**, if the set of vertices is infinite and if each vertex is incident with at least two edges.

Example 3.0.3. We look at some finitely generated groups with symmetric generating set and their Cayley graphs.

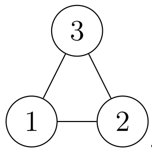
- $\mathbb{Z} =< \pm 1 >$ and CAY($\mathbb{Z}$):= 

- $\mathbb{Z}^2 =< (\pm 1, 0), (0, \pm 1) >$ and CAY($\mathbb{Z}^2$):= 

Definition 3.0.4. An **automorphism** of a graph Γ is a bijection $\rho : \Gamma \to \Gamma$, where we have:

- $\rho(\alpha) = \bar{\alpha}$, $\forall \alpha, \bar{\alpha} \in V$ (in particular we could have $\rho(\alpha) = \alpha$)
- $\forall$e=(v,w) $\in E$ $\rho(e) = (\rho(v), \rho(w)) \in E$.

The **automorphism group** of Γ, denoted by AUT(Γ), is the group of all automorphisms of Γ.

Example 3.0.5. We look at the automorphism group of Γ:= 

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We have AUT(Γ)=S_3=$\{id, (1, 2), (2, 3), (1, 3), (1, 2, 3), (3, 2, 1)\}$.

Definition 3.0.6. Given a group G and a graph Γ=(V, E). An **action of G on** Γ is a homomorphism $\alpha : G \longrightarrow AUT(\Gamma)$, $g \longmapsto \alpha(g)$ and we denote $\alpha(g)v =: g(v)$ for every v $\in$ V. Furthermore we have:

- (hg)(ν)=h(gν), $\forall$ h,g $\in G$ and $\forall\ \nu \in \Gamma$
- id(ν)=ν, $\forall \nu \in \Gamma$.

The action is called **free** if we have: $g(v) = v$ then $g = id$ for some $v \in \Gamma$.

Now we will concentrate on the free group $\mathbb{F}_2$, we will see how its Cayley graph looks and we will see an automorphism-group of the tree $\mathcal{T}_4$.

Proposition 3.0.7. *The Cayley graph of* $\mathbb{F}_2$ *is the tree* $\mathcal{T}_4$.

Proof. Let $\mathbb{F}_2$ be generated by S=$\{x, y\}$. We consider T=$S \cup S^{-1}$=$\{x, y, x^{-1}, y^{-1}\}$ which is also a generating set of our free group (see proof of Theorem 2.0.6) and which is symmetric. We can immediately conclude that the Cayley graph is connected, because the vertices of CAY($\mathbb{F}_2$) are the elements of the group, and every element of $\mathbb{F}_2$ is generated by T. Furthermore we have that CAY($\mathbb{F}_2$) is a 4-regular graph, because T consists of four elements and thus every vertex has four neighbours. Now we need to show that the CAY($\mathbb{F}_2$) is a tree:
Let's suppose that is not true, which means there exists a cycle in CAY($\mathbb{F}_2$). Let $\gamma = v_o, ..., v_n$ be a closed path in CAY($\mathbb{F}_2$), with $v_0 = v_n$. This implies by the definition of the Cayley graph that there is a reduced word $\omega \in \mathbb{F}_2$ of length n, such that:

$$v_0\omega = v_n = v_0.$$

But by definition of a free group, there is no such word ω except the empty word, which has length zero. This is a contradiction, because by definition a path has at least length one. Thus, there is no such cycle in CAY($\mathbb{F}_2$).
Thus CAY($\mathbb{F}_2$) is a connected 4-regular tree, i.e. $\mathcal{T}_4$ (see Figure 1.1). □

Before seeing another propriety of the free group $\mathbb{F}_2$ we need to introduce some facts about the automorphisms of trees.

Definition 3.0.8. Given a d-regular tree $\mathcal{T}$

- A geodesic between two vertices $\alpha, \beta \in \mathcal{T}$ is the unique path which connects both vertices.
- Given a vertex $\nu \in \mathcal{T}$ and $\xi \in \delta\mathcal{T}$, where $\delta\mathcal{T}$ is the boundary of the tree (you can imagine it as the set of the endpoints of $\mathcal{T}$ in infinity). We define the **infinite geodesic ray** as the unique geodesic between ν and ξ.
- Given $\xi_1, \xi_2 \in \delta\mathcal{T}$. A **biinfinite geodesic** is the unique geodesic between ξ_1 and ξ_2.

Definition 3.0.9. Let $\mathcal{T}$ be a tree. An automorphisms $\gamma \in$ AUT($\mathcal{T}$) is :

- **elliptic** if γ fixes a vertex v, i.e. $\gamma(v) = v$,
- **an inversion** if γ fixes an edge e=(v,w) but switched the ends, i.e. $\gamma(e) = \gamma(v, w) = (w, v) = (v, w) = e$ and $\gamma(v) = w$, $\gamma(w) = v$,
- **hyperbolic** of step k along a biinfinite geodesic (ξ_1, ξ_2), if there exists a biinfinite geodesic $(\xi_1, \xi_2) = ...v_{-1}v_0v_1...$ in $\mathcal{T}$ such that $\gamma(v_i) = v_{i+k}$ for all i.

Proposition 3.0.10. *Every automorphism of a tree is either elliptic, or an inversion or hyperbolic.*

Corollary 3.0.11. *The free group $\mathbb{F}_2$ is a subgroup of the automorphism group of $\mathcal{T}_4$.*

Now we will see, what this corollary means, and how $\mathbb{F}_2$ acts on its "own" Cayley graph. We have seen that the Cayley graph of $\mathbb{F}_2$ is the tree $\mathcal{T}_4$. Furthermore we have defined the action of a group on a graph, which will be useful now to construct a specific action. Let S=$\{x, y\}$ be the generating set of $\mathbb{F}_2$. Let us take the generator x and a vertex ν in $\mathcal{T}_4$, which corresponds to a freely reduced word in $\mathbb{F}_2$, since CAY($\mathbb{F}_2$)=$\mathcal{T}_4$. We develop the given vertex so we can write it as a concatenation of a letter ($\beta \in S \cup S^{-1}$) and a sub-word ($\nu' \in \mathbb{F}_2$), which gives us: $\nu = \beta\nu'$. We define the action $\alpha : \mathbb{F}_2 \longrightarrow \text{AUT}(\mathcal{T}_4)$ for the generator x by:

$$\alpha(x)v = x * \nu = x * \beta\nu' = \begin{cases} x\nu \text{ if } x^{-1} \neq \beta \\ \nu' \text{ if } x^{-1} = \beta. \end{cases} \tag{3.1}$$

This action of x on the graph $\mathcal{T}_4$ is like a “shift to the right“, or in other words a translation on the right. To imagine that, look at the line $H := \{x^n \,|\, n \in \mathbb{Z}\}$, which is a biinfinite geodesic. The action $\alpha(x)$ on this geodesic transforms every element $x^k \in H$ into $\alpha(x)(x^k) = xx^k = x^{k+1} \in H$. So if we compare this to the definition seen above, we conclude that this action defines an automorphism of $\mathcal{T}_4$, precisely we have a hyperbolic automorphism along H. We could enlarge the above action by looking at the action $\alpha(y)$ for the generator $y \in S$ and we would come to a similar conclusion, namely that the action of y on the tree $\mathcal{T}_4$ is a "shift to the top". In other words, $\alpha(y)$ is hyperbolic along the biinfinite geodesic $\{y^n \,|\, n \in \mathbb{Z}\}$.

After seeing some properties of the free group $\mathbb{F}_2$, we want to respond to the question if our free group has any subgroup and if yes, which are those.

Proposition 3.0.12. *There is a finite index subgroup of $\mathbb{F}_2$ that is a free group of rank 3.*

Proof. We start by defining a subset of $\mathbb{F}_2$ as follows:

$$H := \{\gamma \in \mathbb{F}_2 |\, |\gamma| = 2n, n \in \mathbb{N}\}$$

H is the subset of $\mathbb{F}_2$, which contains only words of even length. Our aim is now to show that:

1. H is a soubgroup

2. the index $|\mathbb{F}_2$:H$|$ is finite

3. H is a free group of rank 3.

We start with (1): It is trivial that H is closed under taking inverses, because we have noticed above that the length of a word is equal to the length of its inverse:

$$\omega \in H \Rightarrow \omega^{-1} \in H, \text{ because } |\omega| = |\omega^{-1}|.$$

Now we need to show that the concatenation of to words of even length is also of even length.

By the equivalence relation which induced the freely reduced words, we have seen, that we remove two neighbouring elements (letters) in a word if they are inverses to one another. This implies that we cancel only an even number of letters of the concatenation while reducing it. So we have:

$\alpha, \beta \in H \Rightarrow |\alpha\beta| \leq |\alpha| + |\beta|$ and because we cancel only an even number of letters we get that $|\alpha\beta|$ is even. Thus, $\alpha\beta \in$ H.

We have showed, that the group H is a subgroup of $\mathbb{F}_2$.
Let proof (2): Every element in $\mathbb{F}_2$ consists either of an even or an odd number of letters. So half of the elements in $\mathbb{F}_2$ are in our even subgroup H. Thus:

$$|\mathbb{F}_2\text{:H}|=2<\infty.$$

Finally, we need to show (3). We need to show that H is generated by a set S_H consisting of 3 elements. We will see just the beginning of the argument, the rest would be very similar.

We define $S_H := \{x^2, xy, xy^{-1}\}$ and we want to show by induction on the length of the words, that it generates H. That means every element of H is a concatenation of elements in $S_H \cup S_H^{-1}$.

Start with $\delta \in H$, $|\delta|$=2, which is a reduced word. So we can write δ in many ways, some of them are listed above, and show that it is always generated by elements of $S_H \cup S_H^{-1}$ and therefore is an element of M:=$\{S_H \cup S_H^{-1}\}$*:

- $\delta := x^2 \in$M
- $\delta := y^2 = yy = yx^{-1}xy = (xy^{-1})^{-1}xy \in$M
- $\delta := xy \in$M
- $\delta := yx = yx^{-1}x^2 = (xy^{-1})^{-1}x^2 \in$M
- etc.

You continue until you showed it for all words of length 2. Then continue in the same manner with the induction. □

Chapter 4

Free group homomorphisms

The aim of this chapter is to define a group homomorphism from a free group to a given group G, which will help us to prove two important results regarding the presentations of groups.

Theorem 4.0.1. *Let G be any group and let* $\{g_1, ..., g_n\}$ *be a set of elements in G, which are not necessarily distinct or non-trivial. Let* $S = \{x_1, ..., x_n\}$ *be a basis for a free group* $\mathbb{F}_n$*. Then there is a unique group homomorphism* $\phi : \mathbb{F}_n \to G$ *where* $\phi(x_i) = g_i$*.*

Proof. We refer here to the proof of the Universal Property 2.0.7 of Jan Wey's talk. □

Corollary 4.0.2. *Any two free groups of rank n are isomorphic.*

Proof. Let $\mathbb{G}$ and $\mathbb{H}$ be free groups of rank n. Let $S_G = \{g_1, ..., g_n\}$ be the basis of $\mathbb{G}$ and $S_H = \{h_1, ..., h_n\}$ be the basis of $\mathbb{H}$. From Theorem 4.0.1 and the Universal Property we have, that there exist unique homomorphisms:

$$\varphi : \mathbb{G} \to \mathbb{H},\ g_i \mapsto h_i,$$
$$\psi : \mathbb{H} \to \mathbb{G},\ h_i \mapsto g_i.$$

By composing both homomorphisms, we can see:

$$\varphi \circ \psi = id_{\mathbb{G}},$$
$$\psi \circ \varphi = id_{\mathbb{H}}.$$

Thus φ and ψ are bijections. Thus $\mathbb{G} \approx \mathbb{H}$. □

Corollary 4.0.3. *If G is generated by n elements, then G is a quotient of* $\mathbb{F}_n$

Proof. See Jan Wey's talk, proof of Corollary 2.0.8. □

Chapter 5

Free groups and actions on trees

In this chapter, we will see two very important results, which characterize free groups. Before that, we need to define one important concept, which will be used in the proof of the first result.

Definition 5.0.1. Let G be a group acting on a connected tree $\mathcal{T}$. The subtree $\mathcal{F} \subset \mathcal{T}$ is called a **fundamental domain** for the action of G on the tree $\mathcal{T}$, if the following propriety is satisfied:

$$\forall x \in \mathcal{T} \ \exists g \in G \text{ with } gx \in \mathcal{F}.$$

Theorem 5.0.2. *A finitely generated group G is free if and only if it acts freely on a tree.*

Proof. Outline:
$\Rightarrow$: Let G be a free group with basis S:=$\{x_1, ..., x_n\}$. For this direction we refer to the proof of Proposition 3.0.7, since the idea of showing that the Cayley graph is a tree, is identical.
Furthermore we have that the action of G on CAY(G) is free, because if $g(v) = v$ for some $g \in$G and some $v \in$CAY(G), we can conclude that $g = \emptyset$ in G, since G is free. Thus, the action is free.
$\Leftarrow$: Here we aim to use the Ping-Pong Lemma.
Let us assume that G acts freely on a 2n-regualr tree $\mathcal{T}$. We would like to construct the fundamental domain of this action. This will help us find the symmetric subset of G and the connected subtrees of $\mathcal{T}$ which are needed for the Ping-Pong Lemma.
We want to construct the fundamental domain of the free action of G on the tree $\mathcal{T}$. Let start by fixing $v \in \mathcal{T}$ and by considering subtrees $\mathcal{K} \subset \mathcal{T}$, which satisfy:

- $v \in \mathcal{K}$
- if a,b $\in \mathcal{K}$ are distinct, then we have that $ga \neq b \, \forall g \in G$.

We can find at least one maximal subtree with the above proprieties and we denote it as CORE. Here we claim that the G-orbit of the CORE contains all vertices of $\mathcal{T}$ but not all edges of the tree are in the G-orbit of the CORE. The goal now is to add those missing edges to the CORE in order to obtain a fundamental domain, such that the G-orbit of the fundamental domain gives

us the whole tree $\mathcal{T}$.
We continue by adding those missing edges to the G-orbit of the CORE, denoted by Im(CORE), such that we obtain the required property of the G-orbit of the fundamental domain. For each edge $e \in \mathcal{T}$ with $e \cap$Im(CORE)$=\emptyset$, define h_e as the closed half-edge of e that intersects the CORE. Finally we define the fundamental domain $\mathcal{F}$ as the union of the CORE and those half-edges h_e, i.e. $\mathcal{F} =$CORE $\bigcup\limits_{e \cap Im(CORE)=\emptyset} h_e$.

The aim now is to find a symmetric subset of G, such that we can later use the Ping-Pong Lemma: So we continue by choosing for every half-edge $h_e \in \mathcal{F}$ an element $g_e \in G$ such that $\mathcal{F} \cap g_e \cdot \mathcal{F}$ is the middle point of the edge e (or in other words, it is the end of the half-edge h_e). We denote by $\mathcal{S}$ the union of all g_e. By the definition of g_e we have that $\mathcal{F} \cap g_e \cdot \mathcal{F}$ is a midpoint, but in the mean time $g_e^{-1} \cdot \mathcal{F} \cap \mathcal{F}$ is also a midpoint, because in our infinite tree $\mathcal{T}$ every edge has a "converse"-edge, so we conclude: if $g_e \in \mathcal{S}$, then $g_e^{-1} \in \mathcal{S}$. We can then redefine this set by $\mathcal{S} = S \cup S^{-1}$, since every element of $\mathcal{S}$ can be paired with its inverse. Thus we have constructed the symmetric subset of G, but we still need to find the connected subtrees of $\mathcal{T}$:

We define for every $h_e \in \mathcal{F}$ the tree $\mathcal{T}_e$ as the maximal subtree in $\mathcal{T}\backslash$CORE, which contains a vertex incident to the edge e (here we encourage the reader to look up the definition of the half-edges h_e seen before in the proof). Or equivalently we could define those trees $\mathcal{T}_e$ as follows: Fix a vertex $v \in$ CORE and define for each $g_e \in \mathcal{S}$ the tree $\mathcal{T}_e$ as the subtree of $\mathcal{T}\backslash$CORE, such that for every $v' \in \mathcal{T}\backslash$CORE the unique path from v to v' contains the half-edge h_e. The conclusion at this point in the proof is, that we have constructed the symmetric subset of G and the connected subtrees of $\mathcal{T}$, which we need for using the Ping-Pong Lemma.

To finish the proof, we would need to show the two following necessary proprieties of the Ping-Pong Lemma, which is left to the reader (otherwise look it up in [1]). For $p \in$ CORE we want to show:

- $g_e(p) \in \mathcal{T}_e$ for all $g_e \in S$,
- $g_{e_1}(\mathcal{T}_{e_2}) \subset \mathcal{T}_{e_1}$ for all $g_{e_1} \in \mathcal{S}$ and $g_{e_2} \in \mathcal{S} \setminus \{g_{e_1}\}$.

After having shown the previsous two points, we can finally conclude by the Ping-Pong Lemma that G is a free group with basis S. □

Corollary 5.0.3. *(Nielsen-Schreier) Every subgroup of a free group is free.*

Proof. Given a free group $\mathbb{F}$. Let H $\subset \mathbb{F}$ be a subgroup. We know and have seen, that the Cayley graph of $\mathbb{F}$ is a tree and that the action of $\mathbb{F}$ on its Cayley graph is free. Thus, the subgroup H also acts freely on the Cayley graph of $\mathbb{F}$. We can now conclude by Theorem 5.0.2, because H acts freely on CAY($\mathbb{F}$), that H is free. □

Bibliography

[1] John Meier; *Groups, Graphs and Trees*, London Mathematical Society, 2008.

[2] Andre Haefliger, Raghavan Narasimhan; *Essays on Topology and Related Topics*, Springer Berlin Heidelberg, 1970.

[3] Frank Herrlich; *Gruppen und Graphen*, Notes, Karlsruher Institut für Technologie, 2007.

[4] Corran Webster; *Harmonic Analysis and Representation of the Automorphism Groups of Homogeneous Trees*, Honours Thesis, University of New South Wales, 1991.

[5] Jan Wey; *Free groups I*, Pro-seminar manuscript, University of Fribourg, 2017.